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Non-local gauge fields and Zilch

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Abstract. The Yang–Mills gauge theory is applied to the space–time invariance groups responsible for the conservation of Zilch, with the result that non-local interactions are forced upon us. In this case, if there is also an internal symmetry group present, the Yang–Mills trick yields gauge fields which transform according to not only the regular representation of the group, but also all the representations whose coupling to the original fields is allowed by non-zero Clebsch–Gordan coefficients.

1. Introduction

The Yang–Mills (1954) method of introducing interactions by means of a generalized gauge principle is now formally well understood since the general discussions of Utiyama (1956), Kibble (1961) and many others (e.g. Bludman 1955, Shima 1964, Lubkin 1963, Sokolik and Konopleva 1965, Loos 1965, Ikeda and Miyachi 1956).

The basic idea is simply to make the parameters of an invariance group functions of space–time and to restore the resulting loss of invariance by the introduction of compensating fields. For example, coordinate-dependent phase transformations lead to the electromagnetic field, while the gravitational field emerges when coordinate-dependent Lorentz transformations are considered.

It thus seems reasonable, whenever we have an invariance group, to enquire what sort of interaction is introduced as a compensation if we make the parameters functions of position. If the interaction so introduced does not occur in nature, further doubts would be cast on the logical status of the gauge principle but not, of course, upon its interest and utility.

In this paper we wish to discuss these questions for the invariance group (or groups) responsible for the Zilch-type conserved quantities discussed first by Lipkin (1964) and elaborated by, for example, Kibble (1965), Fairlie (1965), Fradkin (1965) and Steudel (1965).

Roughly speaking, we may say that the generators of the appropriate groups are products of the generators of the inhomogeneous Lorentz group (Steudel 1966, 1965, Dowker 1968). Typically, invariance under the transformation of a scalar field $\varphi(x)$

$$\varphi(x) \to \varphi(x) + \delta_0\varphi(x)$$

with

$$\delta_0\varphi(x) = a^{\mu\nu\rho}\partial_\mu\partial_\nu\partial_\rho\varphi(x)$$

yields the conserved Zilch-like quantity (for massless fields)

$$S^\mu_{\rho\sigma\nu} = \partial_\rho\partial_\sigma\partial_\nu\varphi\partial^\mu\varphi - 3\delta^\mu_{(\rho}(\partial_\nu\partial_{\sigma)}\partial_\alpha\varphi\partial^\alpha\varphi - \partial_\sigma\partial_\alpha\varphi\partial_{\nu)}\partial^\alpha\varphi + \partial_\alpha\varphi\partial_\nu\partial_{\sigma)}\partial^\alpha\varphi)$$

where the permutation sum conecrns just ρ, ν and σ.

2. Basic ideas and calculation

We shall restrict ourselves to scalar fields and to invariance groups generated by products of displacement operators. Thus, under the displacement $x^\mu \to x^\mu + a^\mu$, φ suffers the change $\delta_0\varphi$ given by

$$\delta_0\varphi(x) = a^\mu\partial_\mu\varphi(x) + \tfrac{1}{2}a^\mu a^\nu\partial_\mu\partial_\nu\varphi(x) + \tfrac{1}{6}a^\mu a^\nu a^\rho\partial_\mu\partial_\nu\partial_\rho\varphi(x) + \dots . \quad (1)$$

Invariance of the theory under this transformation leads to the conserved energy–momentum tensor. Now, with Steudel (1966), we generalize (1) to

$$\delta_0\varphi(x) = a^\mu\partial_\mu\varphi(x) + a^{\mu\nu}\partial_\mu\partial_\nu\varphi(x) + a^{\mu\nu\rho}\partial_\mu\partial_\nu\partial_\rho\varphi(x) + \dots \quad (2)$$

where the $a^{\mu \cdots}$ are independent parameters. It can be shown, and it will be proved later, that the usual scalar (Klein–Gordon) theory is invariant under the 'odd derivative' terms in this transformation and a series of conserved Zilch-like quantities thus results.

In accordance with our previous remarks, we now make the $a^{\mu \cdots}$ functions of space–time and investigate how to restore the lost invariance.

Instead of using the infinite series (2) we shall write, equivalently,

$$\delta_0 \varphi(x) = \int \Lambda(x, x')\varphi(x')\,\mathrm{d}x' = -\int \Lambda(x, x-x')\varphi(x-x')\mathrm{d}x' \tag{3}$$

the integration being over all space–time. A Taylor series expansion of $\varphi(x-x')$ in the integrand of (3) gives expressions for the $a^{\mu \cdots}$ as moments of Λ.

Any particular term or set of terms in the series (2) can be eliminated by an appropriate choice of Λ. Thus the even derivatives disappear if we have the antisymmetry condition

$$\Lambda(x, x-x') = -\Lambda(x, x+x'). \tag{4}$$

If the $a^{\mu \cdots}$ are constants, $\Lambda(x, x')$ is a function of $x - x'$ only.

It is seen that the analysis is a generalization of the usual one (e.g. Utiyama 1956) to the non-local case. Or we could say that the indices of the 'internal' symmetry group have become continuous and are the space–time coordinates themselves.

We should point out that Toro (1965) has already discussed a non-local extension of Utiyama's formalism, but we find ourselves a little puzzled by his treatment and so will proceed *ab initio*.

The technique of restoring the invariance amounts, partly, to a search for a suitable 'covariant derivative'. This comes about in the following way.

If the theory is invariant under (2), or (3), with the $a^{\mu \cdots}$ constant, this will no longer be true if the $a^{\mu \cdots}$ are functions of space–time since $\delta_0(\partial_\mu \varphi)$ now has extra terms resulting from the derivative acting on the $a^{\mu \cdots}$. Thus from (3)

$$\delta_0(\partial_\mu \varphi) = \partial_\mu \delta_0 \varphi = -\int \partial_\mu^x \varphi(x-x')\Lambda(x, x-x')\,\mathrm{d}x' - \int \partial_\mu^x \Lambda(x, x-x')\varphi(x-x')\,\mathrm{d}x'. \tag{5}$$

If $\Lambda(x, x')$ is a function of $x-x'$ only, the second term on the right-hand side of (5) is zero. This suggests that we look for an expression, say $\nabla_\mu \varphi$, which reduces to $\partial_\mu \varphi$ in the absence of any compensation and which transforms, under (3), like $\partial_\mu \varphi$ when the $a^{\mu \cdots}$ are constants. Part of the lost invariance will then be restored on replacing $\partial_\mu \varphi$ by $\nabla_\mu \varphi$ in the Lagrangian. We shall see this explicitly shortly.

We expect that $\nabla_\mu \varphi$ will look like

$$\nabla_\mu \varphi(x) = \int K_\mu(x, x')\varphi(x')\,\mathrm{d}x' \tag{6}$$

and we shall use the transformation requirement

$$\delta_0(\nabla_\mu \varphi(x)) = + \int \Lambda(x, x')\,\nabla_\mu \varphi(x')\,\mathrm{d}x' \tag{7}$$

to determine how the non-local gauge field $K_\mu(x, x')$ transforms under (3).

We shall now employ a continuous matrix notation and rewrite (3), (6) and (7) as

$$\delta_0 \varphi = \Lambda \varphi \tag{8}$$

$$\delta_0(\nabla_\mu \varphi) = \Lambda K_\mu \varphi = \Lambda \nabla_\mu \varphi \tag{9}$$

$$\nabla_\mu \varphi = K_\mu \varphi. \tag{10}$$

From (10) and (8) we have

$$\delta_0(\nabla_\mu \varphi) = K_\mu \Lambda \varphi + \delta_0 K_\mu \varphi.$$

Comparing this with (9) we see that we must have

$$\delta_0 K_\mu = [\Lambda, K_\mu]_- \tag{11}$$

which is, in form, a familiar result from the Yang–Mills theory.

Using this matrix notation, let us check the earlier statements about the invariance under the odd derivatives of (2) of the usual scalar field theory. The action for this latter is given by

$$S = \tfrac{1}{2}(\partial_\mu \tilde{\varphi} \partial^\mu \varphi - m^2 \tilde{\varphi}\varphi)$$

and the change in the action under (8) is

$$\delta_0 S = \tfrac{1}{2}\{\partial_\mu \tilde{\varphi}(\tilde{\Lambda}+\Lambda)\partial^\mu \varphi - m^2 \tilde{\varphi}(\tilde{\Lambda}+\Lambda)\varphi\}$$

where

$$\tilde{\Lambda}(x, x') \equiv \Lambda(x', x).$$

Since, in the case where the $a^{\mu\cdots}$ are constants, $\Lambda(x, x')$ is a function of $x - x'$ only, the condition (4) which eliminates the even derivatives from (2) means that $\tilde{\Lambda} = -\Lambda$, and so $\delta_0 S$ is zero as stated.

Now, in the general case when the $a^{\mu\cdots}$ are functions we see that S is no longer invariant under (8), partly because of the more complicated transformation of $\partial_\mu \varphi$ and partly because Λ is, in general, no longer antisymmetric, which means that $\tilde{\varphi}\varphi$ is not now invariant. We have found the remedy for the first difficulty—we replace $\partial_\mu \varphi$ by $\nabla_\mu \varphi$, and we now give the remedy for the second. Let us introduce a matrix $M = \tilde{M}$ which transforms such that $\tilde{\varphi} M \varphi$ is invariant. This condition implies that the change in M under (8) is

$$\delta_0 M = -(\tilde{\Lambda} M + M\Lambda). \tag{12}$$

We then consider the theory described by the action

$$S_\varphi = \tfrac{1}{2}(\nabla_\mu \tilde{\varphi} M \nabla^\mu \varphi - m^2 \tilde{\varphi} M \varphi) \tag{13}$$

and clearly now $\delta_0 S_\varphi$ vanishes. The invariance has been restored by introducing the compensating fields K_μ and M.

Quite generally, we should expect K_μ and M to be independent, but we wish to go further and introduce a restriction which connects them.

We recognize that M is rather like a metric which can be used for raising and lowering indices or for changing covariant into contravariant quantities. As is usual in theories with a metric, we shall require the operations of 'raising and lowering' to commute with covariant differentiation. This means that the covariant derivative of M must vanish. To determine what this implies we make use of the fact that $\tilde{\varphi} M \varphi$ is invariant and constant: then

$$\nabla_\mu(\tilde{\varphi} M \varphi) = 0$$

whence, on the assumption that ∇_μ is a distributive operation,

$$\nabla_\mu M = -(MK_\mu + \tilde{K}_\mu M)$$

or

$$\tilde{K}_\mu M = -MK_\mu. \tag{14}$$

This is a relation connecting M and K_μ. It is easily checked that it is consistent with (11) and (12).

We may in fact develop this formalism a little more. The quantity $\tilde{\varphi} M \nabla_\mu \varphi$ is also invariant and constant; thus

$$\nabla_\nu(\tilde{\varphi} M \nabla_\mu \varphi) = 0$$

which implies that

$$\nabla_\mu K_\nu = [K_\mu, K_\nu] = 2K_{[\mu} K_{\nu]}. \tag{15}$$

3. The action for the gauge field

Pursuing the analogy with Yang–Mills theory we now enquire for a possible action principle describing the K_μ field itself. To this end let us consider the quantity $R_{\mu\nu}$ defined by

$$R_{\mu\nu} = K_{[\mu} K_{\nu]}.$$

Under (8) the change in $R_{\mu\nu}$ is given by

$$\delta_0 R_{\mu\nu} = [\Lambda, R_{\mu\nu}]$$

and a suitably invariant action is

$$S_k = \mathrm{Tr}(R_{\mu\nu}R^{\mu\nu}).$$

The action principle

$$\frac{\delta S_k}{\delta \tilde{K}_\mu} = 0$$

then yields the equations of motion of the free K_μ field

$$K^\mu K_\mu K_\nu + K_\nu K_\mu K^\mu - 2K^\mu K_\nu K_\mu = 0 \tag{16}$$

which, using (15), can be written in the more conventional way

$$\nabla^\mu \nabla_\mu K_\nu = 0.$$

4. The coupled equations of motion

The total action S is equal to $\lambda S_\varphi + S_K$ and the coupled equations of motion are given by the variational equations

$$\frac{\delta S}{\delta \varphi} = 0, \qquad \frac{\delta S}{\delta \tilde{K}_\mu} = 0$$

or

$$\nabla_\mu \nabla^\mu \varphi = K^\mu K_\mu \varphi = -m^2 \varphi \tag{17}$$

and

$$\nabla^\mu \nabla_\mu K_\nu = \tfrac{1}{2}\lambda(\varphi \otimes \tilde{\varphi} M K_\nu + K_\nu \varphi \otimes \tilde{\varphi} M) \tag{18}$$

where we have used (14) and (17) in the variation of S_φ with respect to $\tilde{K}_\nu$ in order to eliminate the quantity $\delta M/\delta \tilde{K}_\nu$. The direct product symbol $\otimes$ refers to the continuous labels. The trace of the left-hand side of (18) is zero, as can be seen explicitly from the left-hand side of (16), and it is easily checked that the trace of the right-hand side of (18) is also zero, for consistency, i.e. we have

$$\tilde{\varphi} M K_\nu \varphi = \bar{\varphi} K_\nu \varphi = 0.$$

For convenience we have here defined the 'conjugate' quantity $\bar{\varphi}$ by

$$\bar{\varphi} = \tilde{\varphi} M.$$

Its covariant derivative is given by

$$\nabla_\mu \bar{\varphi} = \nabla_\mu(\tilde{\varphi} M) = \nabla_\mu \tilde{\varphi} M = \tilde{\varphi} \tilde{K}_\mu M = -\bar{\varphi} K_\mu \tag{19}$$

and so, from (17), it satisfies the equation of motion

$$\nabla_\mu \nabla^\mu \bar{\varphi} = \bar{\varphi} K_\mu K^\mu = -m^2 \bar{\varphi}. \tag{20}$$

With the aid of (19) and (20) it is easy to prove that the right-hand side of (18), which we shall denote by λJ_ν, has a vanishing covariant divergence, i.e.

$$\nabla^\nu J_\nu = 0. \tag{21}$$

5. Example

As an example, let us consider the case when just the first term of (2) exists. Symbolically we then have for Λ the expression

$$\Lambda = \mathrm{i} a^\mu P_\mu \tag{22}$$

where P_μ is the momentum operator. The matrix element of P_μ is given by

$$\langle x | \mathrm{i} P_\mu | x' \rangle = \partial_\mu \delta(x - x').$$

Similarly in (22) the quantity a^μ is a diagonal operator (or matrix) with elements

$$\langle x|a^\mu|x'\rangle = a^\mu(x)\delta(x-x').$$

From (11) we find for the corresponding change in K_μ the form

$$\delta_0 K_\mu = \mathrm{i}a^\rho P_\rho K_\mu - \mathrm{i}K_\mu a^\rho P_\rho. \tag{23}$$

Now, in general, K_μ is a polynomial in $\mathrm{i}P_\nu$ and we require that $\delta_0 K_\mu$ should be a polynomial of the same kind and degree, otherwise the theory would not be covariant. This is, in fact, just the point of the present paper, for if we choose any term, but the first, of (2) it is impossible to satisfy this requirement with K_μ a polynomial of *finite* degree. In other words, non-locality is forced upon us *even if* the transformation on φ can be expressed in local fashion, i.e. with a finite number of derivatives.

The only exception to this is the example under discussion in the present paragraph. To see this, let us rewrite (23) using the rule

$$\mathrm{i}[P_\mu, A] = \overset{+}{\partial}_\mu A \qquad (\simeq \partial_\mu A,\ A \text{ diagonal})$$

where the quantity $\overset{+}{\partial}_\mu A$ has matrix elements

$$\langle x|\overset{+}{\partial}_\mu A|x'\rangle = \partial_\mu A(x,x') + \partial'_\mu A(x,x').$$

We have

$$\delta_0 K_\mu = a^\rho \overset{+}{\partial}_\rho K_\mu + [a^\rho, K_\mu]\mathrm{i}P_\rho$$

and so, if we choose

$$K_\mu = \mathrm{i}h_\mu^{\cdot\nu} P_\nu, \qquad h_\mu^{\cdot\nu} \text{ diagonal} \tag{24}$$

we see that we can write $\delta_0 K_\mu$ as

$$\delta_0 K_\mu = \mathrm{i}(a^\rho \partial_\rho h_\mu^{\nu} - h_\mu^{\cdot\rho}\partial_\rho a^\nu)P_\nu = \mathrm{i}\delta_0 h_\mu^{\cdot\nu} P_\nu$$

which is of the same form as K_μ.

Since the transformation in question can be generated from space–time translations, we expect the formalism to reduce to that given by Kibble (1961) for the same case. To check that this is so, let us calculate M from the condition (14) and (24). Thus, since P_μ is antisymmetric,

$$P_\nu h_\mu^{\cdot\nu} M = M h_\mu^{\cdot\nu} P_\nu$$

or

$$\partial_\nu h_\mu^{\cdot\nu} M + h_\mu^{\cdot\nu}\overset{+}{\partial}_\nu M + \mathrm{i}[h_\mu^{\cdot\nu}, M]P_\nu = 0.$$

A solution to this equation for M can be found if we take M to be diagonal, for then $h_\mu^{\cdot\nu}$ and M commute and $\overset{+}{\partial}_\nu M \equiv \partial_\nu M$. In this case we have

$$\partial_\nu h_\mu^{\nu} M = -h_\mu^{\cdot\nu}\partial_\nu M \qquad \text{or} \qquad \partial_\nu \ln M = -h^{\mu}_{\cdot\nu}\partial_\sigma h_\mu^{\cdot\sigma} \tag{25}$$

where the quantity $h_\mu^{\cdot\nu}$ is the inverse of $h_\mu^{\cdot\nu}$, i.e.

$$h^{\mu}_{\cdot\rho} h_\mu^{\cdot\sigma} = \delta_\rho^{\cdot\sigma}.$$

We compare equation (25) with equation (6.5) of Kibble (1961) which, allowing for the different notation, yields

$$\Gamma^{\lambda}_{\nu\lambda} = \partial_\nu \ln\sqrt{-g} = \partial_\nu \ln \det|h| = -h^{\mu}_{\cdot\nu}\partial_\lambda h_\mu^{\cdot\lambda}$$

if all considerations of angular momentum (spin and orbital) are ignored, i.e. the 'stroke' derivative is just an ordinary derivative. Thus we can identify M with $\det|h|$ as expected.

Since we are dealing with only translations, it is not surprising that we do not recover the full content of Kibble's analysis.

6. The Bianchi identity: conservation laws

The covariance of the theory under the 'gauge' transformations (3) yields

$$\int \frac{\delta S}{\delta \tilde{K}_\mu(x, y)} \delta \tilde{K}_\mu(x, y)\, dx\, dy = 0$$

or

$$\mathrm{Tr}(S^\mu K_\mu \Lambda - S^\mu \Lambda K_\mu) \equiv \mathrm{Tr}\{(S^\mu K_\mu - K_\mu S^\mu)\Lambda\} = 0$$

$$S^\mu = \frac{\delta S}{\delta \tilde{K}_\mu}$$

whence, since Λ is an arbitrary function, the 'Bianchi' identity

$$-S^\mu K_\mu + K_\mu S^\mu = \nabla_\mu S^\mu = 0.$$

From the equation of motion (18) we then obtain the 'conservation' law (21).

From this non-local law we can derive *local* conservation laws. As an example we consider the free-field scalar theory. In this case K_μ is just $\mathrm{i}P_\mu$, and we have explicitly for J_ν the expression

$$J_\nu(x, x') = \tfrac{1}{2}\{\varphi(x)\partial'_\nu\varphi(x') - \partial_\nu\varphi(x)\varphi(x')\}.$$

Equation (21) now reads

$$(\partial_\nu + \partial'_\nu)J^\nu(x, x') = 0 \tag{26}$$

and this can be verified directly using the equations of motion for φ. Local conservation laws can be derived from equations like (26) by putting x equal to x'. Thus for any operator A we have

$$\langle \overset{+}{\partial}_\nu A\rangle \equiv \langle x|\overset{+}{\partial}_\nu A|x\rangle = \partial_\nu\langle A\rangle = \partial_\nu A(x, x)$$

which equations define the symbol $\langle\ \rangle$. Similar considerations to these occur in the work of Green (1949) and Born (1949) on the statistical matrix and the theory of reciprocity.

J^μ is antisymmetric and so $\langle J^\mu\rangle$ vanishes. However, we can obtain non-trivial results by first multiplying by powers of P^ν. Thus for the symmetric quantity

$$J^{\mu\nu} = \mathrm{i}[P^\mu, J^\nu]_+ = \tilde{J}^{\mu\nu}$$

the bracket $[\ ]_+$ signifying the anticommutator, we have

$$\partial_\nu\langle J^{\mu\nu}\rangle = 0.$$

Explicitly written out

$$\langle J^{\mu\nu}\rangle = -\tfrac{1}{2}\varphi\overleftrightarrow{\partial^\mu}\,\overleftrightarrow{\partial^\nu}\,\varphi$$

where $\varphi\overleftrightarrow{\partial^\mu}\varphi \equiv \varphi\partial^\mu\varphi - \partial^\mu\varphi\varphi$. Thus $J^{\mu\nu}$ is symmetric on the tensor indices. It is one of a class of conserved tensors discussed by Kibble (1965) and differs from the usual canonical energy–momentum tensor by an explicit divergence, yielding the same integrated total energy and momentum.

The reason why we do not obtain the actual canonical tensor is partly because we have treated surface terms in a very cavalier way. Assumption (14) is partly equivalent to throwing away a divergence in the integrand of the action. This is best appreciated in the free-field case, $K_\mu = P_\mu$ and $M = 1$, when (14) is equivalent to the statement that the momentum operator is antisymmetric, $\tilde{P}_\mu = -P_\mu$, and can act either forwards or backwards (see Dirac 1947, § 22). The other, related and more basic, reason is our neglect of angular momentum, as mentioned in the previous paragraph. We refer again to the free-field scalar case. The standard method of finding the energy momentum tensor is to write the theory in curvilinear coordinates, i.e.

$$S = \tfrac{1}{2}\int (\varphi_{\|\mu}\varphi^{\|\mu} - m^2\varphi^2)\sqrt{-g}\, d^4x \tag{27}$$

and then to use the identification

$$\frac{\delta S}{\delta g_{\mu\nu}} = -\tfrac{1}{2}T^{\mu\nu}\sqrt{-g}$$

where $g_{\mu\nu}$ is the curvilinear metric and $g = \det\|g_{\mu\nu}\|$. The usual canonical energy momentum tensor† is obtained from $T^{\mu\nu}$ by replacing $g_{\mu\nu}$ by $\eta_{\mu\nu} = \mathrm{diag}(-1, -1, -1, 1)$. Now, by throwing away a divergence, the action (27) is equivalent to S', where

$$S' = -\tfrac{1}{2}\int (\varphi\varphi^{\|\mu}_{.\mu} + m^2\varphi^2)\sqrt{-g}\,\mathrm{d}^4x.$$

Varying S' with respect to $g_{\mu\nu}$ still yields the same $T^{\mu\nu}$. However, in the non-local theory considered in the present paper, since we do not have the full geometric interpretation in terms of a space with metric $g_{\mu\nu}$, we should not expect the action analogous to S', i.e. $-\frac{1}{2}(\varphi K^\mu K_\mu \varphi + m^2\bar{\varphi}\varphi)$, to yield the canonical tensor on suitable variation. We might expect, and indeed we do find, that the tensor actually obtained gives the same integrated energy–momentum vector, since it is this latter which generates translations and these are included in our theory, we hope correctly.

If we pursue the procedure of multiplying by powers of P^μ in order to derive other conservation laws, it is reasonably clear that we shall obtain tensors of the form

$$Q^{(m)(n)\nu} = \partial^{(m)}\varphi\overleftrightarrow{\partial}^{\nu}\partial^{(n)}\varphi$$

$$\partial_\nu Q^{(m)(n)\nu} = 0.$$

These tensors are of the general type

$$\varphi_1\overleftrightarrow{\partial}^{\nu}\varphi_2$$

where φ_1 and φ_2 are any solutions of the Klein–Gordon equation.

We attribute the non-appearance of Steudel's tensor $S^{\mu}_{\rho\sigma\nu}$ to our neglect of angular momentum. Presumably it gives the same integrated quantities as the tensor

$$\varphi\overleftrightarrow{\partial}^{\mu}\,\overleftrightarrow{\partial}^{\nu}\,\overleftrightarrow{\partial}^{\alpha}\,\overleftrightarrow{\partial}^{\beta}\varphi.‡$$

7. Introduction of an internal symmetry

An interesting situation arises if we extend the previous calculation to cover the case when φ belongs to some representation of an internal symmetry group. All that is necessary is to add a discrete internal index to the continuous space–time one. The fundamental equations will then be unchanged in form. Following the standard procedure (e.g. Utiyama 1956), we shall write

$$\Lambda = \Lambda_i T^i, \qquad [T^i, T^j] = f^{ij}_k T^k$$

where the T^i are the generators of the internal Lie groups. For generality we have in mind the GL(n, C) and U(n) groups.

For the change in K_μ we find from (11) the expression

$$\delta_0 K_\mu = [\Lambda_i T^i, K_\mu].$$

In contrast to the usual situation it is not possible, in general, to expand $\delta_0 K_\mu$ also in terms of just the generators for, if we do write

$$K_\mu = K_{\mu i}T^i \tag{28}$$

we find

$$\delta_0 K_\mu = \tfrac{1}{2}[T^i, T^j]\{\Lambda_i, K_{\mu j}\} + \tfrac{1}{2}\{T^i, T^j\}[\Lambda_i, K_{\mu j}] \tag{29}$$

† Canonical only in the scalar field case.

‡ This has been confirmed.

where the curly brackets stand for the anti-commutator. Normally, i.e. in the local case, the second term on the right-hand side of (29) is proportional to the generators, but not so here since Λ_i and $K_{\mu i}$ are (continuous) matrices. Thus, in general, $\delta_0 K_\mu$ is not a linear combination of generators, which means that the formalism is not consistent, i.e. covariant. An exception is when φ belongs to the defining (fundamental) representation of the group for, in this representation, the generators actually do form a complete set. In this case the introduced gauge field belongs to the regular representation of the group (e.g. Glashow and Gell-Mann 1961).

When φ belongs to a general representation a suitable complete set is provided by the irreducible and symmetrized products of the generators, the largest product being of $[J]$ generators, where $[J]$ is the dimension of the representation to which φ belongs (the 'J' representation, J standing for the complete set of quantum numbers defining the representation).

In general, all these symmetrized products will be needed for K_μ. Thus, if as suggested by (29), we add a term proportional to $\{T^i, T^j\}$ to (28), then the new $\delta_0 K_\mu$ will contain a symmetrized product of *three* generators and so on.

Algebraically these products are a bit cumbersome and a more convenient technique is provided by some Clebsch–Gordan analysis. Thus we could employ the quantities $u_q^{(k)}$ introduced† by Racah (1951) which are, essentially, operators in 'angular momentum space' with matrix elements given by a $3j$ symbol, i.e.

$$\langle jm|u_q^{(k)}|jm'\rangle = \begin{pmatrix} j & q & m' \\ m & k & j \end{pmatrix}, \qquad k = 0, 1, \dots 2j.$$

These matrices form a complete set for $(2j+1)$-square matrices, in particular for the generators of $U(2j+1)$ in the defining representation, as Racah noted. The complete algebra of the $u_q^{(k)}$ is known in terms of Racah coefficients, and all we have to do is to expand T^i and K_μ in terms of the $u_q^{(k)}$, j being given by $2j+1 = [J]$. Another possibility would be to generalize the $u_q^{(k)}$ (i.e. the $3j$ symbols) to the internal group in question (see Wigner 1938). Then these generalized $3j$ symbols,

$$\langle JM|U_Q^{(K)}|JM'\rangle = \begin{pmatrix} J & Q & M' \\ M & K & J \end{pmatrix}$$

would again span the space of $(2j+1)$-square matrices if $2j+1 = [J]$. In particular, the generators T^i are obtained when the K representation is the regular one R, i.e.‡

$$T^i \propto U_i^{(R)}$$

We should then need to expand only K_μ in terms of the $U_Q^{(K)}$.

This method has the advantage of explicitly showing the symmetry group content of the gauge field K_μ. For $U(2)$, the original Yang–Mills group, these two methods coincide. The details will be presented at another time.

The type of interaction that we should obtain was, of course, obvious from the start, for example

$$\sum_K g_K \begin{pmatrix} J & Q & M' \\ M & K & J \end{pmatrix} \partial^\mu \bar{\varphi}_{(J)}^M K_{\mu Q}^{(K)} \varphi_{M'}^{(J)}.$$

8. Conclusion

When we try to extend the Yang–Mills gauge theory to the invariance groups responsible for the conservation of Zilch-like tensors, we are forced immediately into a discussion of non-local field theory, the physical import of which is, at present, somewhat hazy§. Further, our theory is incomplete since only space–time translations are considered.

† We have retained Racah's notation. Really we ought to write $u^q_{(k)}$.

‡ If the unit matrix is included amongst T_i, then we also need $U_0^{(0)} \propto 1$.

§ It should be stated here that Fairlie's (1965) discussion of Zilch shows that non-locality comes in somewhere.

Instead of considering odd multiples of P_μ, i.e. space–time transformation generators, we could apply the Yang–Mills trick to invariance groups whose generators are *products* of the generators of *internal* symmetry groups. Thus, for the case of the homogeneous Lorentz group, the generalized covariant derivative for a field transforming according to the $(j, 0)$ representation, say, would take the form

$$\partial_\mu + a^i_\mu J_i + a^{(ijk)}_\mu J_i J_j J_k + \ldots + a^{(i\cdots k)}_\mu \underbrace{J_i \ldots J_k}_{2j-1 \text{ factors}} \tag{30}$$

where the J_i are the usual angular momentum matrices. The second term in (30) represents the gravitational field (spin 2). The other terms presumably indicate an interaction with fields of increasing spin, up to a maximum of $2j$. Of course, we should expect to have a covariant derivative like (30) if we had included angular momentum right from the beginning. The necessity for such a form is apparent from the discussion of § 7.

References

Bludman, S. A., 1955, *Phys. Rev.*, **100,** 372–5.
Born, M., 1949, *Rev. Mod. Phys.*, **21,** 463–79.
Dirac, P. A. M., 1947, *Quantum Mechanics*, 3rd edn (Oxford: Clarendon Press).
Dowker, J. S., 1968, *J. Phys.* A (*Proc. Phys. Soc.*), [2], **1,** 277–9.
Fairlie, D. B., 1965, *Nuovo Cim.*, **37,** 897–904.
Fradkin, D. M., 1965, *J. Math. Phys.*, **6,** 879–90.
Glashow, S. L., and Gell-Mann, M., 1961, *Ann. Phys.*, N.Y., **15,** 437–60.
Green, H. L., 1949, *Proc. R. Soc.* A, **197,** 73–89.
Ikeda, M., and Miyachi, Y., 1956, *Prog. Theor. Phys., Japan*, **16,** 537–47.
Kibble, T. W. B., 1961, *J. Math. Phys.*, **2,** 212–21.
—— 1965, *J. Math. Phys.*, **6,** 1022–6.
Lipkin, D. M., 1964, *J. Math. Phys.*, **5,** 696–700.
Loos, H. G., 1965, *Nucl. Phys.*, **72,** 677–91.
Lubkin, E., 1963, *Ann. Phys.*, N.Y., **23,** 233–83.
Racah, G., 1951, *Group Theory and Spectroscopy* (Princeton: Institute for Advanced Study).
Shima, K., 1964, *Prog. Theor. Phys., Japan*, **31,** 139–50.
Sokolik, H. A., and Konopleva, N. P., 1965, *Nucl. Phys.*, **72,** 667–76.
Steudel, H., 1965, *Nuovo Cim.*, **39,** 395–8.
—— 1966, *Z. Naturf.*, **21a,** 1826–8.
Toro, T. I., 1965, *C.R. Acad. Sci., Paris*, **261,** 2593–6.
Utiyama, R., 1956, *Phys. Rev.*, **101,** 1597–607.
Wigner, E. P., 1938, *On the Matrices which Reduce the Kronecker Products of Representations of S.R. Groups* (unpublished manuscript reprinted in *Quantum theory of angular momentum*, Eds L. C. Biedenharn and H. van Dam, 1965 (New York: Academic Press)).
Yang, C. N., and Mills, R. L., 1954, *Phys. Rev.*, **96,** 191–5.